

# On the critical slowing down exponents of mode coupling theory

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A method is provided to compute the parameter exponent  $\lambda$  yielding the dynamic exponents of critical slowing down in mode coupling theory. It is independent from the dynamic approach and based on the formulation of an effective static field theory. Expressions of  $\lambda$  in terms of third order coefficients of the action expansion or, equivalently, in term of six point cumulants are provided. Applications are reported to a number of mean-field models: with hard and soft variables and both fully-connected and dilute interactions. Comparisons with existing results for Potts glass model, ROM, hard and soft-spin Sherrington-Kirkpatrick and  $p$ -spin models are presented.

In the framework of glassy mean-field (MF) models with quenched or built-in disorder, it is well known that the dynamics of models whose glassy phase is consistently described by a Replica Symmetry Breaking (RSB) solution with a finite number of breaking's displays critical slowing down and a dynamic transition [1–4]. The equations governing the relaxation dynamics down to the dynamic critical temperature  $T_d$  are those pertaining to the schematic mode-coupling-theory (MCT) developed in the context of supercooled liquids [5–9]. The “guide observable” is the correlation function  $C(t)$ , i.e., the overlap between a given initial equilibrium configuration of the dynamics and the configuration at time  $t$ . In the discontinuous one step RSB case near  $T_d$  the relaxation of  $C(t)$  is a two step process in which the system spends a large amount of time, algebraically diverging as  $T \rightarrow T_d$ , around a plateau value  $q_{EA}$ . Models with a discontinuous dynamic transition include, e.g., the spin-glass (SG)  $p$ -spin model with either spherical or Ising spins, the Potts glass model and the Random Orthogonal Model (ROM). MCT predicts that two exponents control the whole dynamics. Exponent  $a$  governs the early  $\beta$  regime as  $C(t)$  approaches the plateau value with a power law  $C(t) \approx q_{EA} + c_a/t^a$ , while exponent  $b$  identifies the early  $\alpha$  regime as  $C(t) = q_{EA} - c_b t^b$ , when the system starts relaxing towards equilibrium [8]. An important prediction of MCT is the following relationship between the decay exponents and their mutual relationship to the so-called “exponent parameter”  $\lambda$ :

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \lambda \quad (1)$$

In case of a continuous transition, there is no dynamic arrest and no  $b$  exponent is defined. Well known instances are, e.g., the paramagnet to full-RSB SG transition along the de Almeida Thouless (dAT) line in mean-field SG models, either fully connected [10, 11] or on random graphs [12], as well as the SG transition in Potts models with  $p \leq 4$ , both fully connected [13] and on the Bethe lattice of any connectivity [14], and in the  $p$ -spin spherical model with large external magnetic field [3]. Even though Eq. (1) is usually assumed as the correct relationship between exponents  $a$  and  $b$ , the exponent parameter  $\lambda$ , apart from schematic MCT cases, is simply

considered a tunable parameter, without a specific connection to any physical observable. Here we show how to unveil this connection in full generality working within a “static-driven” effective theory of dynamics: we will put forward an independent formulation of  $\lambda$  and apply it to some paradigmatic SG models. In general, the analytic treatment of the dynamics is more complicated than the statics and only a few models have been studied so far: the soft-spin Sherrington-Kirkpatrick (SK) model [10, 11], schematic MCT's [6], soft-spin  $p$ -spin and Potts glass models, for which the connection with MCT was first identified [2]. This prompted to consider the spherical  $p$ -spin SG [3, 15] in all details as a MF structural glass, cf., e.g., Ref. [16], even in the off-equilibrium regime below  $T_d$  [17]. In these cases dynamics is explicitly solved and  $\lambda$  exactly computed. In particular, one finds that *it is not universal* and depends on model and external parameters. On the other hand, its computation becomes difficult when we consider more complicated MF systems and finite-dimensional ones.

*Static definition of the exponent parameter  $\lambda$ .* Similarly to the static transition also the dynamic one can be located as the critical point of an appropriate replicated Gibbs free energy  $\Gamma$  [18–20]. This is a function of the replicated dynamic variables (e.g. spins  $\sigma$ ), Legendre transform of the replicated free energy  $\Phi$

$$\Gamma[\delta Q_{ab}] = \Phi[\Lambda_{ab}] + \sum_{ab} \Lambda_{ab} \delta Q_{ab}, \quad \frac{\partial \Phi}{\partial \Lambda} = \delta Q_{ab} \quad (2)$$

with respect to a conjugated field  $\Lambda$ , function of the overlap matrix. We used the short-hand  $\delta Q_{ab} = \sigma_a \sigma_b - q$ ;  $q = \langle \sigma_a \sigma_b \rangle$  being the Edwards Anderson parameter, that is the value of the overlap matrix elements for the replica symmetric (RS) solution. Average  $\langle \dots \rangle$  is performed over the proper replicated ensemble [21] [46]. Expanded around the RS critical point  $\Gamma$  reads:

$$\Gamma(\delta Q) = \frac{1}{2} \sum_{(ab), (cd)} \delta Q_{ab} M_{ab,cd} \delta Q_{cd} - \frac{w_1}{6} \text{Tr} \delta Q^3 - \frac{w_2}{6} \sum_{ab} \delta Q_{ab}^3 \quad (3)$$

with  $a, b, c, d = 1, \dots, n$ . We have retained only two of all

cubic coefficients for they are the relevant ones at criticality [22]. The case  $n = 0$  is relevant for the continuous transition [23–25], the case  $n = 1$  for the dynamic discontinuous transition [18–20, 26]. Our main result is:

$$\lambda = \frac{w_2}{w_1} \quad (4)$$

We note that this ratio also yields the breaking point  $x$  in the case of continuous transition from RS to full-RSB [13]. The above result can be obtained in the context of the supersymmetric formulation of the dynamics [27] that is a very convenient way of seeing the connection between equilibrium dynamics and the static replica treatment. Details of the derivation will be given elsewhere [28].

Eq. (4) holds in full generality above the upper critical dimension. In general, we do not have an analytic expression of the Gibbs free energy, e.g., for MF models defined on finite-connectivity random graphs. However,  $\Gamma$  is defined as the Legendre transform of  $\Phi$  and, therefore, its proper vertices can be associated to cumulants of the replicated order parameter. We, thus, face the problem of computing  $\Gamma$  from  $\Phi$  in presence of fields  $\Lambda_{ab}$  coupled to  $1/N \sum_i s_i^a s_i^b$ . The free energy  $\Phi$  needs to be computed at the third order, i.e., dealing with eight RS cumulants  $\omega_{1,\dots,8}$  [22]. The coefficients  $w_1$  and  $w_2$  of the Gibbs potential, cf. Eq. (3), however, are expressed as

$$w_1 = r^3 \omega_1 \quad w_2 = r^3 \omega_2 \quad (5)$$

where  $r$  is the inverse of the SG susceptibility:

$$r = \left[ \frac{1}{N} \sum_{ij} \overline{\langle s_i s_j \rangle_c^2} \right]^{-1} \quad (6)$$

i.e., the replicon, and  $\omega_1$  and  $\omega_2$  can be written in terms of six spin correlations (details elsewhere [28]):

$$\omega_1 = \frac{1}{N} \sum_{ijk} \overline{\langle s_i s_j \rangle_c \langle s_j s_k \rangle_c \langle s_k s_i \rangle_c} \quad (7)$$

$$\omega_2 = \frac{1}{2N} \sum_{ijk} \overline{\langle s_i s_j s_k \rangle_c^2} \quad (8)$$

where  $\overline{\dots}$  means average over the disorder and  $\langle \dots \rangle_c$  denotes the connected thermal average. In case of discontinuous transition it is implicit that different thermal averages in the above expression are all computed within the same state, selected by the initial condition. We underline that while vertices  $w_1$  and  $w_2$  remain finite at  $T_d$ , the corresponding  $\omega$  cumulants diverge as  $r^{-3}$ .

*Method validation and  $\lambda$  computation.* Applying Eq. (4) and computing  $w_1$  and  $w_2$  within the above mentioned static approach, we have tested our prediction on various models. In those few cases where  $\lambda$  is exactly known from the dynamics [3, 11, 35] we verify that its formal expressions are identical. In more complicated systems the dynamic MCT phenomenology has

been studied numerically and estimates of the exponents are available in the literature. These are instances in which Eq. 4 yields actual analytic prediction for  $\lambda$ .

*Fully connected models.* In the following, we consider a family of models with a Hamiltonian of the kind:

$$\mathcal{H} = - \sum_{i < j} J_{ij} \sigma_i \sigma_j - \sum_{p=1}^{\infty} \sqrt{\frac{R^{(p)}}{p!}} \sum_{i_1 < \dots < i_p} K_{i_1, \dots, i_p}^p \sigma_{i_1} \dots \sigma_{i_p}$$

where  $\sigma_i$  are  $N$  Ising spins, or soft/spherical ones. The 2-body interaction matrix is constructed as  $J = \mathcal{O}^T \mathcal{E} \mathcal{O}$ , where  $\mathcal{O}$  is a random  $O(N)$  matrix chosen with the rotational invariant Haar measure, cf., e.g., Ref. [29] and  $\mathcal{E}$  is a diagonal matrix with elements independently chosen from a distribution  $\rho(\varepsilon)$  [30]. In order to ensure the existence of the thermodynamic limit, the support of  $\rho(\varepsilon)$  must be finite and independent of  $N$ . The  $p$ -body interactions  $K^{(p)}$  are i.i.d. Gaussian variables with zero mean and variance  $p!/N^{p-1}$  and  $R^{(p)} = d^p R(x)/dx^p(x)|_{x=0}$  for some real valued function  $R(x)$ . For the MF Ising SG [1, 4, 24], as well as for spherical SG's [15, 31, 32], the general form of the replicated free energy is:

$$-n\beta\Phi = \text{extr}_{Q,\Lambda} \mathcal{S}[Q, \Lambda] \quad (9)$$

$$\mathcal{S}[Q, \Lambda] = \frac{1}{2} \text{Tr} G(\beta Q) + \frac{\beta^2}{2} \sum_{ab} R(Q_{ab}) \quad (10)$$

$$- \frac{1}{2} \text{Tr} Q \Lambda + \ln \text{Tr}_{\{\sigma\}} \mathcal{W}[\Lambda; \{\sigma\}]$$

$$\mathcal{W}[\Lambda; \{\sigma\}] = \exp \left\{ \frac{1}{2} \sum_{a,b} \Lambda_{ab} \sigma_a \sigma_b \right\} \quad (11)$$

where  $G : M_{n \times n} \rightarrow M_{n \times n}$  is a function in the space of  $n \times n$  matrices, formally defined through its power series around zero. Its form depends on the choice of the eigenvalue distribution  $\rho(\varepsilon)$  of the  $\mathcal{E}$  matrix.

Given this effective action  $\mathcal{S}[Q, \Lambda]$ , the saddle point equations in  $\Lambda$  and  $Q$  respectively read

$$Q_{ab} = \langle \sigma_a \sigma_b \rangle_{\mathcal{W}} \quad (12)$$

$$\Lambda_{ab} = \beta [G'(\beta Q)]_{ab} + \beta^2 R'(Q_{ab})$$

that, in the RS Ansatz, become

$$q = \langle m^2(z) \rangle_z; \quad m(z) = \langle \sigma \rangle_{\sigma} \quad (13)$$

$$\Lambda = \frac{\beta}{n} [G'(\beta(1 + (n-1)q)) - G'(\beta(1 - q))] + \beta^2 R'(q)$$

where the weights over which  $\langle \dots \rangle_z$  and  $\langle \dots \rangle_{\sigma}$  are performed depend on  $\sigma$  being Ising, spherical or soft. For the cases of interest here, weights are proportional to the following distributions:

	$\langle \dots \rangle_{\sigma}$	$\langle \dots \rangle_z$
Ising	$e^{z\sigma} [\delta(\sigma+1) + \delta(\sigma-1)]$	$e^{-z^2/(2\Lambda)} \cosh^n(z)$
Spher.	$e^{z\sigma} \exp\{-\sigma^2/[2(1-q)]\}$	$e^{-z^2/(2\Lambda)} e^{n(1-q)z^2/2}$

implying  $m(z) = \tanh(z)$  for Ising and  $(1-q)z$  for spherical spins. To compute the values of  $w_{1,2}$ , cf. Eq. (3), one

$p \rightarrow 2$	2.05	2.2	2.5	3	4	5	6	7	8	9
$\lambda$	.5	.556	.652	.719	.743	.746	.743	.739	.736	.733
$a$	.395	.379	.346	.32	.308	.306	.308	.31	.311	.313

TABLE I: Dynamic exponents in the Ising  $p$  spin model.

has to expand Eq. (10) to third order around the saddle point value. Considering the second order term and imposing that the Hessian determinant vanishes (criticality condition) we obtain

$$\langle\langle(\sigma - m)^2\rangle_\sigma^2\rangle_z = [\beta^2 G''(\beta(1 - q)) + \beta^2 R''(q)]^{-1} \quad (14)$$

Using the above condition and expanding Eq. (12) to second order, we derive the general expressions of  $w$ 's for fully-connected systems:

$$\frac{3!}{\beta^2} w_1 = \frac{\beta}{2} G'''(\beta(1 - q)) + A(q) \langle\langle(\sigma - m)^2\rangle_\sigma^3\rangle_z \quad (15)$$

$$\frac{3!}{\beta^2} w_2 = \frac{R'''(q)}{2} + 2A(q) \langle\langle(\sigma - m)^3\rangle_\sigma^2\rangle_z \quad (16)$$

$$A(q) \equiv \beta^4 [G''(\beta(1 - q)) + R''(q)]^3$$

holding both for  $n = 0$  and  $n = 1$ . They can be used to compute  $\lambda$  in different cases. We now exemplify a few.

For the SK model,  $R(x) = x$ ,  $G(x) = x^2/2$ ,  $\rho(\varepsilon) = \sqrt{4 - \varepsilon^2}/(2\pi)$ , Sompolinsky's result for Ising spins along the dAT line is recovered [47], cf. Eq. 2.61 of Ref. [11].

For the ROM model [4, 30, 33], one has  $R(x) = 0$ ,

$$2G(x) = v_\alpha(x) - 1 + 2(\alpha - 1) \ln \frac{v_\alpha(x) + 2x + 2\alpha - 1}{2\alpha} - \ln \frac{v_\alpha(x) + 1 + 2x(2\alpha - 1)}{2}$$

$$v_\alpha(x) \equiv \sqrt{1 + 4x(2\alpha - 1 + x)}$$

$\rho(\varepsilon) = \alpha\delta(\varepsilon - 1) + (1 - \alpha)\delta(\varepsilon + 1)$ , and the transition is dynamic. MCT dynamics in the ergodic phase has been numerically studied in Ref. [34] for  $\alpha = 13/32$ , where strong finite-size effects are observed and two different estimates for the exponent provided:  $b = .62$ , from the fit of the von Schweidler law, while  $b = .75$ , from the fit of the equilibrium  $\alpha$  relaxation time vs. temperature. Our "static-driven" computation yields  $b = .628$  ( $\lambda = .7077$ ), allowing for a validation of the first numerical estimate.

For Ising  $p$ -spin,  $G = 0$  and  $R(x) = x^p/2$ ,  $\lambda$  values are in Tab. I. For  $p \rightarrow 2$  we retrieve the result of Ref. [2].

For spherical models,  $G = 0$ ,  $R(x) = \sum_r x^r/2$ , the exact analytic form  $\lambda = \lambda''(q_d)(1 - q_d)^3/2$  is retrieved [3, 35, 36], equivalent to the schematic MCT prediction for very long times at criticality, i.e., for  $C(t) = q_d$  [7, 9],  $\Lambda(C(t))$  being the MCT equations memory kernel.

Yet another example is the Potts glass:

$$\mathcal{H} = - \sum_{ij} J_{ij} (p \delta_{\sigma_i \sigma_j} - 1) \quad (17)$$

The model has a discontinuous glass transition for  $p > 4$  [13]. Brangian *et al.* studied the Potts glass with  $p = 10$

by means of Monte-Carlo numerical simulations [37]. Approaching the dynamic transition, finite-size effects turn out to be large, implying that the *plateau* is almost invisible also for very large sizes: this makes the numerical estimation of exponents very difficult. Their interpolation yields  $a = .33 \pm .04$ . For  $p = 10$ , from the expansion of  $\Gamma[\delta Q]$  around  $q = q_d$ , we obtain the exact values  $\lambda = .8053$  and  $a = .2759$  [38], compatible with, though not extremely near to, the numerical estimate.

*Models on diluted random graphs.* To study glassy models on random graphs we set up an apart technical method to analytically compute dynamic exponents. To frame the results, we first recall that there is no single instance of these class of models whose dynamics has been solved explicitly. Eq. (4) would allow to bypass the problem provided one had the replicated action  $\Gamma$ . Unfortunately, the action is hard to compute in diluted models, thus preventing  $w$ 's derivation. We can, however, explicitly compute the related cumulants  $\omega_{1,2}$  at criticality, cf. Eqs. (7)-(8). Details of the derivation will be given elsewhere [38], while here we present a validation of the results for the Viana-Bray model [39] in a field, displaying a continuous transition along the dAT line.

Let us consider a Bethe lattice with connectivity  $c = 4$ , in a field  $h = .7$  at the corresponding critical temperature  $T_c(h) = 0.73536(1)$  [40]. The cumulants ratio yields  $\lambda = 0.461$ . Our analytic prediction for the  $\beta$  decay exponent is, thus,  $a_{th} = .406$ . We, then, numerically study the system by means of both equilibrium and off-equilibrium Monte Carlo simulations. At equilibrium sizes  $2^8 \leq N \leq 2^{13}$  have been probed. Starting from an equilibrated configuration we measure the decay of  $C(t) \equiv \sum_i s_i(0)s_i(t)/N$  with time. For each sample 6 replicas are simulated and 16 uncorrelated measurements taken for each replica. Number of samples: from 1280 ( $N = 2^{13}$ ) to 5120 ( $N = 2^6$ ). Off-equilibrium, the two time  $C(t, t_w)$  is measured. For very long  $t_w$ ,  $C(t, t_w)$  tends to the equilibrium  $C(t)$ . For each sample we take one measure (at the longest simulated  $t_w$ ). The number of samples goes from 1280 ( $L = 2^{20}$ ) to 51200 ( $L = 2^6$ ). Due to finite size effects,  $C(t)$  (at and off-equilibrium) displays a power law behavior only for times smaller than a time scale  $t^*(N)$  that diverges with  $N$ . This makes the estimation of the exponent hard. Therefore, rather than  $C(t)$ , we probe its thermal and disorder fluctuations,

$$\chi_4(t) = N [\overline{C(t)^2} - \overline{C(t)}^2] \quad (18)$$

remaining finite for large  $N$  at finite  $t$ . Using finite-time scaling arguments it can be argued that (i)  $\chi_4(t)$  diverges as  $t^a$  at large times on a scale smaller than  $t^*(N)$ , (ii) the critical region diverges as  $t^*(N) = N^{1/(3a)}$ , (iii) on times scales larger than the critical region the fluctuations scale as  $N^{1/3}$  [41]. As a consequence, if  $a$  has the correct value, the rescaled dynamic critical  $\chi_4(t)/N^{1/3}$  vs.  $t/N^{1/(3a)}$ , should be size independent. We plot it in Fig. 1 with  $a = a_{th} = 0.406$ . Collapse appears excellent both at and off-equilibrium.

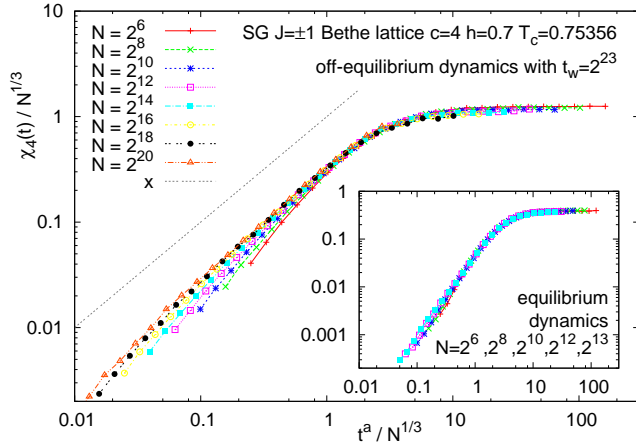


FIG. 1: Rescaled  $\chi_4(t)$  for the Viana-Bray model with  $c = 4$ ,  $h = .7$  and  $T = .75356(1)$ , along the dAT line with  $a = a_{th} = 0.406$ . No fitting interpolation is performed.

Concluding, we have introduced a “static-driven” method to obtain, by means of a replica field theory, the dynamic exponents of the critical slowing down. The method allows to determine the MCT parameter exponent  $\lambda$  as the ratio of coefficients of third order terms of the Gibbs free energy action expanded around the critical point, cf. e.g., Eqs. (15-16). Equivalently,  $\lambda$  is shown to be equal to the ratio between six points cumulants of a theory whose action is the Legendre transformed of the Gibbs free energy, cf. Eqs. (7-8). Indeed, *the dynamical exponents can be associated to the ratio between two physical observables computed within a static framework*. We verified the method’s prediction in various MF models, both on fully-connected and diluted graphs, successfully comparing with previous analytical and numerical results. The method can be exported to any glass models whose Gibbs action is computable or whose six point cumulants can be estimated.

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  - [46] This is equivalent to  $\overline{\langle \dots \rangle_J}$ , where brackets and overline denote, respectively, ensemble (at fixed disorder  $J$ ) and disorder average. When disorder is self-induced this is equivalent to the overall thermal average including nested average over pinning and pinned variables [18, 42–44].
  - [47] To be precise, at difference with the original work we have a Gaussian field  $h_i$ , with  $\overline{h_i} = 0$  and  $\overline{h_i h_j} = \delta_{ij} h^2$ ,

but the two cases can be shown to be equivalent [45].